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Parameters of physically non-homogenous media reconstructed from the eigenfrequencies of their free oscillations

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Abstract. This study presents numerical and asymptotic algorithms for deriving wave characteristics of the free oscillations of a vertically non-homogenous fluid. The bathymetric density distribution is reconstructed from the dispersion curves of free oscillations in an non-homogenous fluid. Measurement-accuracy requirements on the input data are studied in order to obtain a sufficiently accurate bathymetric density of the non-homogenous fluid. These methods are shown to be applicable to identifying geometrical and physical non-homogeneities of rods and elastic layers by their resonant frequencies.

Key words: inverse spectral problem, non-homogeneous medium, parameter reconstruction

1. Introduction

Research on internal gravity waves in the ocean is one of the important challenges of modern oceanology. The investigation of internal waves was significantly stimulated by intensively developing remote sensing of the ocean. The internal waves are manifested on the free surface in the form of sun glitters. From the motion of these glitters, the phase velocity of internal waves can be derived as well as their length. From these parameters, the bathymetric density-field distribution is obtained, thus making it possible to locate anomalous densities (certain submerged objects in the ocean). Examples of such objects are fish shoals, submarines, bathyscaphes, scuba divers, sunken ships, etc.

Similar problems arise for non-destructive testing of building structures when the resonant frequencies of their separate structural components provide the basis for a general conclusion about the density and structure of the entire building unit. In geophysics, such problems are connected with mineral prospecting. In this case, the soil, rock, or water body, are set into vibration, and then the nature of sub-bottom or rock non-homogeneity is interpreted based on surface-vibration measurements.

All the above-mentioned problems fall in the category of inverse problems of continuum mechanics. The present work is devoted to Sturm-Liouville inverse spectral problems, *i.e.*, problems of obtaining a variable coefficient of a Sturm-Liouville differential operator from its eigenvalues.

A survey of publications devoted to Sturm-Liouville inverse problems is given in [1]. According to this survey, the first significant result in this field was obtained by Ambartsumian in 1929 who showed that in the general case with no conditions imposed, the

Sturm-Liouville operator is determined ambiguously. The author suggested employing two spectra of the Sturm-Liouville problem under different boundary conditions. In 1946 this was followed by Borg's first systematic study of the inverse problem for the Sturm-Liouville operator. He showed that the Sturm-Liouville operator is determined from two spectra (under different boundary conditions). Further considerable progress in the theory of inverse problems, as is shown in [1], was attained by Chudov in 1949, Marchenko in 1952, Krein in 1951, Ghelfand and Levitan in 1951, Gasimov in 1964 and Tikhonov in 1963.

The first studies using aerospace radar imaging of an ocean column were performed by the team of the Maritime Institution of Hydrophysics (Ukrainian Academy of Sciences) headed by B.A. Nelepo. The treatise [2] generalizes the most important findings in this field and show results that were obtained experimentally by making use of the Soviet Artificial Earth Satellites (AES) operating in optical and microwave ranges. This study also analyses the detected ocean regions with increased bio-productivity, presents calculations of the thermodynamic parameters of the ocean active layer and analyses the physical principles of ocean remote sensing by AES.

As reported in [2] Grogsky and Kudryavtsev in their studies laid the foundations of a theoretical and numerical determination of the fluid non-homogeneity structure based on the law of internal-wave dispersion, which had been assumed as known. They approximated the Brunt-Vaisala (buoyancy) frequency by a 7-parameter piecewise constant function with given locations of the function discontinuities. The intensities for each of its parts were assumed unknown. The solution of the Sturm-Liouville problem by the finite-difference method is reduced to obtaining the matrix eigenvalues. By varying buoyancy frequency parameters, their values were obtained, for which these frequencies were close to the pre-specified values.

Studies by Govorukhina, Potetunko, Ryndina, Cherkesov and Chuprakov reported in 1989 were also devoted to solving inverse problems of fluid wave motion; see [3]. In their work, the Brunt-Vaisala frequency with a single pycnocline was also approximated by a 7-parameter function, but unlike in publications by Grodsky and Kudryavtsev, the Brunt-Vaisala frequency with a single pycnocline was described by a continuous function and the pycnocline position was considered unknown. Those seven parameters and the pycnocline position were obtained by solving the spectral problem by the finite-difference method and by further applying the minimizing Powel method to find the unknown coefficient values of the finite-difference matrix. The accuracy of the reconstructed functions of the Brunt-Vaisala frequencies for specific examples was evaluated employing the metrics C and L_1.

The same treatise [3] suggested several methods for solving the inverse problems of reconstructing the Brunt-Vaisala frequency. It is obtained with a certain error caused by inaccurate measurement of the eigenfrequency due to instrumental error and faulty *in situ* experimental measurements. Data for a certain region of the World Ocean are presented. For this particular region, a problem of constructing the law of internal-wave dispersion is solved including confidence intervals. Test calculations are performed for some specific examples of the Brunt-Vaisala frequency. The uniqueness conditions of the inverse-problem solution are studied, as well as the effect of the input-data error on the accuracy of the Brunt-Vaisala frequency reconstruction employing the metrics C, L_1, L_2.

Reference [4] presents the results of statistically processed *in situ* measurements of temperature and salinity for a particular region of the World Ocean. The error in the buoyancy frequency is obtained by employing in the Sturm-Liouville problems a variable coefficient of a differential operator. With allowance for this error, the confidence intervals for the eigenfrequencies of the free oscillations of a vertically stratified ocean are found. Stratification parameters are reconstructed while allowing for the error of the frequencies measurement.

In order to reconstruct the frequency, Ryndina [5] writes the buoyancy and the problem solution in terms of a Green's function. The problem is reduced to the solution of a nonlinear integral equation. Parameterisation classes are shown, for which this solution is unique. Examples of non-unique (ambiguous) solutions of the problem are given.

In [6], the Brunt-Vaisala frequency is reconstructed under its piecewise-linear approximation. The problems of the solution uniqueness and accuracy of the considered inverse problems are studied. In [7], the inverse spectral problem of obtaining fluid non-homogeneity is solved. A number of publications in the journal *Inverse Problems* are also devoted to the solution of inverse problems of the Sturm-Liouville type.

The authors of [8, 9] suggest various numerical methods for solving the problem under consideration. Applicability and convergence conditions are presented. The significance of the accuracy of the input data is evaluated. The accuracy of the reconstructed variable potential of the second-order differential equation is studied. Test examples are given to illustrate the application of these methods.

References [10, 11] study the existence and uniqueness of the solution of the inverse problem. Classical Sturm-Liouville equations are considered under various supplementary conditions. Reference [12] is devoted to the asymptotic solution of the inverse Sturm-Liouville problem. The reconstructed potential is taken to be singular. In [13] the inverse spectral problem is solved by reducing it to an integral equation. The analytical solution of the inverse Sturm-Liouville problem is put forward in [14]. Here the authors consider the problem of determining the regular Sturm-Liouville operator from two known spectra. In Borg's statement, they obtain explicit formulae for the solution of the inverse problem.

2. Oceanographically stated problem of defining the law of ocean stratification

In order to study the problem about the propagation of internal gravity waves in the ocean, we employ the model of an ideal non-homogeneous incompressible and thermally non-conductive fluid. As is known, when studying a hydrodynamic process in the ocean, dissipative phenomena, such as viscosity, friction, thermal conductivity, and diffusion, can be neglected. Such a dissipation-free approximation would be justified for the motion on a large-enough scale. In this case the range of time variation is assumed to be limited. Consider the theory of internal waves in an adiabatic approximation. Let us neglect the process of mass and energy exchange in the ocean over the time that waves are generated and propagate.

Consider the equations of motion of a vertically stratified ocean. The coordinate system is assumed to be rotating and fixed to the Earth's surface. The equations are as follows [15, Equations 2.34–2.36, pp. 36]:

$$\rho_0\left(\frac{\partial \bar{V}}{\partial t} + f z^0 \times \bar{V}\right) = -\nabla P - \rho g z^0, \quad \operatorname{div} \bar{V} = 0, \quad \frac{\partial \rho_0}{\partial z} + \rho_0 z \cdot V_z = 0. \tag{1}$$

Here, $\bar{V}$ is the velocity vector in the Cartesian coordinate system, P stands for the deviation of the hydrodynamic pressure from the equilibrium pressure, $f = 2\Omega \sin \varphi$ denotes the Coriolis parameter (Ω is the earth's angular velocity, φ is the latitude), ρ designates the deviation of the fluid density from the equilibrium fluid density ρ_0, g is gravitational acceleration, z^0 is an ort (unit vector) directed along the z-axis (vertically upwards and counter-gravitationally).

At the bottom $z = -H = \text{const}$ the no-flux condition is observed. On the free surface of the ocean the following kinematic and dynamic conditions are fulfilled:

$$V_z|_{z=-H} = 0;\; V_Z|_{z=\zeta} = \frac{d\xi}{dt}, \qquad P(x, y, z, t)|_{z=\xi} = 0. \tag{2}$$

The system (1) and boundary conditions (2) are linearized and the solution is sought in the form of running waves:

$$\{V_x, V_y, V_z, P, \rho, \xi\} = \{U(z), V(z), W(z), P(z), R(z), Z\}e^{i(k_1 x + k_2 y - \omega t)}, \tag{3}$$

where k_1, k_2 are wave numbers and ω is the frequency.

The problem under consideration is reduced to a spectral problem. The amplitude function of the vertical component of the fluid particles velocity is unknown. The equation of this problem are:

$$W''(z) - \frac{\mu(z)}{g} W'(z) + \frac{\mu(z) - \omega^2}{\omega^2 - f^2} k^2 W(z) = 0, \tag{4a}$$

$$W'(0) = \frac{gk^2}{\omega^2 - f^2} W(0), \quad W(-H) = 0; \qquad W(0) = 0, \quad W(-H) = 0. \tag{4b, c}$$

Here, $\boldsymbol{\mu}(z) = g\boldsymbol{\rho}_0'/\boldsymbol{\rho}_0$ is the squared buoyancy frequency, $k^2 = k_1^2 + k_2^2$, $W(z)$ is the amplitude function of the vertical component of the velocity of the fluid particles. The conditions (4b) are full boundary conditions. They do not distinguish between surface waves and internal waves. The conditions (4c) correspond to the "rigid lid" approximation. They filter out internal waves from the surface waves.

3. Derivation of frequency equations by a power-series method

Consider Problem (4) in the Boussinesq approximation. According to this approximation, the term $W'(z)\,\boldsymbol{\mu}(z)/g$ in the equation is omitted, and Problem (4) assumes the following form:

$$W''(z) + \frac{\mu(z) - \omega^2}{\omega^2 - f^2} k^2 W(z) = 0, \tag{5a}$$

$$W'(0) = \frac{gk^2}{\omega^2 - f^2} W(0), \quad W(-H) = 0; \qquad W(0) = 0, \quad W(-H) = 0. \tag{5b, c}$$

Let us introduce a dimensionless variable $z = -H\zeta$ and define $\mu(z) = \tilde{\mu}(\zeta)$, $W(z) = \tilde{W}(\zeta)$, $kH = \tilde{k}$. Henceforth the tilda "$\sim$" will be omitted. By substituting ζ for z, we obtain the following statements of this problem:

$$W''(z) + \frac{\mu(z)}{g} W'(z) + \frac{\mu\,(z) - \omega^2}{\omega^2 - f^2} k^2 W\,(z) = 0, \tag{6a}$$

$$W'(0) = -\frac{gk^2}{\omega^2 - f^2} W(0), \quad W(1) = 0; \qquad W(0) = 0, \quad W(1) = 0. \tag{6b, c}$$

In the Boussinesq approximation:

$$W''(z)+\frac{\mu(z)-\omega^2}{\omega^2-f^2}k^2W(z)=0, \tag{7a}$$

$$W'(0)=-\frac{gk^2}{\omega^2-f^2}W(0), \quad W(1)=0; \qquad W(0)=0, \quad W(1)=0. \tag{7b, c}$$

The solution of Problem (7b) is obtained by expanding the sought $W(z)=\sum_{i=0}^{\infty}C_i z^i$ and the function $\mu(z)=\sum_{i=0}^{\infty}\mu_i z^i$ into a power series. Since the equations obtained are to be satisfied for every z, the coefficient of every z-power will be zero. Thus, we obtain a recurrent system of linear equations in C_i. From the boundary equation at the surface we find $C_1=-C_0 gk^2/(\omega^2-f^2)$. On using the bottom conditions, we find the dispersion equation relating oscillation frequencies ω and wave numbers k with coefficients μ_i:

$$F=1+\sum_{i=1}^{N}r_i=0, \quad r_1=-\lambda g, \quad r_2=-a/2, \quad r_3=-(\lambda\mu_1+ar_1)/6,$$

$$r_4=-(r_2a+[\mu_2+\mu_1r_1]\lambda)/12, \quad a=\lambda\mu_0+b, \quad b=-\omega^2\lambda, \quad \lambda=\frac{k^2}{\omega^2-f^2},$$

$$r_m=-\frac{1}{m(m-1)}\left[\lambda\left(\sum_{j=1}^{m-3}\mu_j r_{m-2-j}+\mu_{m-2}\right)+r_{m-2}a\right], \quad m\geq 5. \tag{8}$$

4. Solution of the inverse problem for the frequency equation. Power-series method

Consider the case when the function $\mu(z)$ is a square parabola $\boldsymbol{\mu}(z)=-a_0^2z^2+a_1z+a_2$. The values of the coefficients a_0, a_1, a_2 are assumed to be known. We set the values of ω and derive the values of k from the frequency equation. For the inverse problem, using the known pairs $\{\omega^2, k^2\}$, we reconstruct the values of the parameters $\mu(z)$. We now substitute in the frequency equation the number of pairs $\{\omega^2, k^2\}$ equal to the number of parameters $\mu(z)$. This yields a system of nonlinear equations. This system was solved by Newton's method. Consider the problem of non-uniqueness of the obtained parameters of the function $\mu(z)$. Let us look at the frequency equation in the parameter a_1 for fixed values of a_0 and a_2. Let us plot the function

$$\boldsymbol{\Phi}=\sum_i F^2\left(k_i^2, \boldsymbol{\omega}_i^2, a_0, a_1, a_2\right).$$

Figure 1 shows the graphs of the function. Line 1 corresponds to the values of $\{\omega^2, k^2\}$ only for the first dispersion curve. Line 2 corresponds to the values of $\{\omega^2, k^2\}$ only for the second dispersion curve. Line 3 corresponds to the values of $\{\omega^2, k^2\}$ for both dispersion curves. Let us analyse the graphs. On using the pairs $\{\omega^2, k^2\}$ for the first or the second dispersion curves only, the function Φ each time has two minima. In plotting the graph using the values of $\{\omega^2, k^2\}$ simultaneously from both dispersion curves, only a single minimum remains at the exact value of a_1. Making use of the $\{\omega^2, k^2\}$ values taken from different dispersion curves ensures uniqueness.

This is one of the possible supplementary conditions for the unique reconstruction of the Sturm-Liouville operator. Herewith, the calculation accuracy depends on the distances between the spectral numbers corresponding either to various curves or to different boundary conditions.

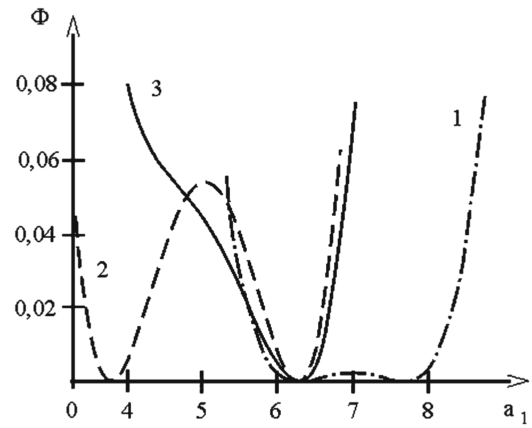

Figure 1. The graphs of the function Φ.

Table 1 shows the reconstructed parameters of the function $\mu(z)$ with various numbers of significant digits used in setting the values of the pairs $\{\omega^2, k^2\}$. The exact values of the parameters are: $a_0 = 6{\cdot}25, a_1 = 6{\cdot}25, a_2 = 1$.

5. Asymptotic construction of the problem solution

If the buoyancy-frequency function shows a large gradient, the power-series method cannot be applied. In this case, the solution of the problem is constructed by means of asymptotic formulae using the WKB (Wentzel-Kramers-Brillouin) method and a parabolic cylinder (Weber) function. By replacing the variable $z = \xi\, t + \beta; \xi = \sqrt[4]{(\omega^2 - f^2)/(4k^2 a_0^2)}, \beta = a_1/(2a_0)$, we reduce the initial-boundary-value problem (7a, b) to the following one ($a_0 \equiv a$):

$$W''(t) + \left(p + \frac{1}{2} - \frac{t^2}{4}\right) W(t) = 0, \quad p = \frac{ak}{2\sqrt{\omega^2 - f^2}} \left(\frac{a_1^2}{4a^4} + \frac{a_2 - \omega^2}{a^2}\right) - \frac{1}{2},$$
$$\frac{1}{\xi}\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{gk^2}{\omega^2 - f^2} W, \quad t = -\frac{a_1}{2a^2\xi}, \quad W\left(\frac{a_1}{2a^2\xi}\right) = 0. \tag{9}$$

5.1. Building up the solution using parabolic cylinder functions

The exact solution of the last equation is found by making use of parabolic cylinder functions [16, pp. 124–126]:

$$W_D = C_1 D_p(t) + C_2 D_p(-t). \tag{10}$$

Table 1. Reconstructed parameters of the function $\mu(z)$.

Number of significant digits	Reconstructed value (reconstruction error)		
	a_2	a_1	a_0
4	1·038 (3·8%)	6·277 (0·43%)	6·448 (3·17%)
3	1·057 (5·7%)	6·306 (0·9%)	6·553 (4·8%)
2	0·7464 (25·3%)	6·19 (0·95%)	5·178 (17·1%)

The integral approximation of parabolic cylinder functions has the following form [16]:

$$D_p(t) = \sqrt{\frac{2}{\pi}} e^{t^2/4} \int_0^\infty e^{-\xi^2/2} \xi^p \cos\left(t\xi - \frac{p\pi}{2}\right) d\xi, \quad \mathfrak{Re}\, p > -1. \tag{11}$$

Several terms of the asymptotic expansion have been obtained by the method of steepest descent at $|t| \to \infty, |p| \to \infty$ in the range $t^2/|p| < 4$.

According to [17, pp. 52–54], the method of steepest descent runs as follows. For large values of the parameter τ, the value of the integral $\int_C e^{\tau\varphi(z)} f(z)\, dz$ is determined by the part of the integration path C, on which $\mathfrak{Re}(\varphi(z))$ is large compared with the values on the remaining part of the C. The path C is distorted in such a way that the contour should go through the saddlepoint z_0 where $\varphi'(z_0) = 0$ and $\mathfrak{Im}\varphi(z) = \text{const}$ on this contour. The first term of the derived asymptotic expansion has the form:

$$\begin{aligned} D_p(t) &= \gamma \cos\left(\frac{t}{4}\sqrt{-t^2 + 4p + 2} - \frac{p\pi}{2} + \left[p + \frac{1}{2}\right] \arcsin \frac{t}{2\sqrt{p + \frac{1}{2}}}\right) \\ &\quad + O\left(\frac{1}{\sqrt{t^2 + |p|}}\right), \quad \gamma = \frac{2e^{-p/2 - 1/4}\left(p + 1/2\right)^{p/4}}{\sqrt[4]{-t^2 + 4p + 2}}. \end{aligned} \tag{12}$$

By substituting asymptotic form (12) in (10) and satisfying the boundary conditions (9), we obtain the solutions of the Problem (7a, b) for large values of k.

5.2. Problem solution constructed by the WKB method

Let us construct the solution of the Problem (7a, b) by the WKB method. The WKB solutions of Equation (7) have the form [18, pp. 281–282]:

$$\begin{aligned} &W_{WKB} = W_1 + W_2, \quad W_{1,2} = \frac{C_{1,2}}{\sqrt{\psi(z)}} \exp\left(\pm i\lambda \int_0^z \psi(\xi)\, d\xi\right), \\ &\lambda = \frac{k^2}{\omega^2 - f^2}, \quad \lambda \to \infty, \quad \omega^2 > f^2, \quad \psi(z) = \sqrt{-a^2z^2 + a_1 z + a_2 - \omega^2}, \quad \psi^2(z) > 0. \end{aligned} \tag{13}$$

Calculate the integrals. This yields an expression for solving Equation (7). This expression fully coincides with the solution resulting from (10), (12).

$$\begin{aligned} &W_{BKE} = \frac{1}{\sqrt[4]{-a^2z^2 + a_1 z + a_2 - \omega^2}} (C_1 \sin\alpha + C_2 \cos\alpha), \\ &\alpha = \lambda\left(\frac{\left(2a^2z - a_1\right)\sqrt{-a^2z^2 + a_1 z + a_2 - \omega^2}}{4a^2} - \frac{4a^2\left(a_2 - \omega^2\right) + a_1^2}{8a^3} \arcsin \frac{a_1 - 2a^2 z}{\sqrt{4a^2\left(a_2 - \omega^2\right) + a_1^2}}\right). \end{aligned} \tag{14}$$

Estimate the error of the WKB solution having the following form:

$$\begin{aligned} &\|V\| \le \frac{b_1 C}{b\,(b\lambda - b_1)}, \text{ where } V(z) = W(z) - W_{WKB}(z), \quad C = \text{const} > 0, \\ &b = \min\left\{a_2,\ -a^2 + a_1 + a_2 - \omega^2\right\}, \quad b_1 = \frac{\left|-8a^2\left(a_2 - \omega^2\right) - 2a_1^2\right|}{\min\{a_2^2, (-a^2 + a_1 + a_2 - \omega^2)^2\}}. \end{aligned} \tag{15}$$

This formula determines the limitations of the applicability of the asymptotic solution constructed using the WKB method for large values of k.

The WKB solution (14) and the solution constructed using the first term of the asymptotics of a parabolic cylinder (12) coincide. By satisfying the boundary conditions we obtain the frequency equation. Introduce the function composed of a sum of square frequency equations for various ω, k. These values are found while solving the spectral problem (7a, b). Solve the inverse problem of reconstructing the parameters of fluid stratification under the parabolic profile of the function $\mu(z)$. On using the value of ω, k with three significant digits, nonhomogeneity parameters are calculated with an accuracy of up to 10%. On setting the input data with two significant digits, we attain a reconstruction accuracy of 15%.

5.3. Asymptotic construction of the solution for a fine-structure case

By fine structure we understand here the presence of a large gradient of the function $\boldsymbol{\mu}(z)$ in the vicinity of one or several points. Such a singularity of the function $\boldsymbol{\mu}(z)$ can be simulated by representing it in the form:

$$\begin{aligned}&\mu(z) = A_0 + A_1\alpha_1\left(\sqrt{\pi}\right)^{-1} \mathrm{e}^{-\alpha_1^2(z+z_1)^2}, \quad A_0,\ A_1,\ \alpha_1 - \text{const}, \quad z \in [-H, 0],\\ &A_1 = AH, \quad \alpha = \alpha_1 H.\end{aligned} \tag{16}$$

The parameter A_0 characterizes the mean stratification. The parameter A_1 reflects the jump in the squared buoyancy frequency, the value z_1 is the pycnocline depth ($z_1 > 0$), and $1/\alpha$ is the pycnocline width.

Solve the considered problem (5a, c) in the "rigid lid" approximation. The axis z is assumed to be counter-gravity. The solution is sought in the following form [3, pp. 123–124]:

$$\begin{aligned}&W^- = C_1 \exp\left(\mathrm{i}\beta z + \int_0^z y^-(\xi)\,\mathrm{d}\xi\right) + C_2 \exp\left(-\mathrm{i}\beta z + \int_0^z y_*^-(\xi)\,\mathrm{d}\xi\right), \quad -z_1 \le z \le 0,\\ &W^+ = C_3 \exp\left(\mathrm{i}\beta(z+H) + \int_{-H}^z y^+(\xi)\xi\right) + C_4 \exp\left(-\mathrm{i}\beta(z+H) + \int_{-H}^z y_*^+(\xi)\,\mathrm{d}\xi\right),\end{aligned} \tag{17}$$

$-H \le z \le -z_1, \quad \beta^2 = \dfrac{A_0 - \omega^2}{\omega^2 - f^2} k^2,\ y^{\pm}, y_*^{\pm}$ are complex conjugate.

Satisfy the boundary conditions at $z = 0$ and $z = -H$ and the continuity conditions of the solutions $W^-(z_1) = W^+(z_1)$, $\left(W^-(z_1)\right)' = \left(W^+(z_1)\right)'$. This yields a system of equations that are homogeneous with respect to the constants C_1, C_2, C_3, C_4. On equating this system determinant to zero, we obtain a dispersion equation for the initial problem. For $\alpha \to \infty$ (to an accuracy of up to $\varepsilon^2 = 1/\alpha^2$) it has the following form:

$$\begin{aligned}&\left[1 + \varepsilon \frac{A\left((\beta H)^2 + (kH)^2\right)}{\sqrt{\pi}\left(A_0 - f^2\right)}\right](\beta H)\sin(\beta H) + A\frac{(\beta H)^2 + (kH)^2}{A_0 - f^2}\\ &\quad\times\left[1 + \varepsilon\left(1 - \frac{\sqrt{2}}{2}\right)\frac{A}{\sqrt{\pi}}\frac{(\beta H)^2 + (kH)^2}{A_0 - f^2}\right] \times \sin\beta z_1 \sin\left(\beta\left(z_1 - H\right)\right) = 0.\end{aligned} \tag{19}$$

In the limiting case $\varepsilon = 0$, Equation (19) converges to the following:

$$\beta H \sin(\beta H) = \frac{(\beta H)^2 + (kH)^2}{A_0 - f^2} A \sin\left((H - z_1)\beta\right)\sin\beta z_1. \tag{20}$$

The roots of Equation (20) are asymptotics at $\varepsilon \to 0$ of the roots of Equation (19). The asymptotics of the roots of Equation (20) for large values of n or m have the following form:

$$\beta_n H = U_n + \delta_n, \quad \beta_m H = V_m + \delta_m, \quad n \gg 1,\ m \gg 1; \quad n,\ m \in Z;$$

$$U_n = \frac{n\pi}{1-q}, \quad V_m = \frac{m\pi}{q}, \quad q = \frac{z_1}{H}, \quad M = \frac{A_0 - f^2}{A}, \tag{21}$$

$$\delta_n = \frac{MU_n \sin U_n}{(-1)^n (1-q)\left[(kH)^2 + U_n^2\right]\sin(qU_n) - M(\sin U_n + U_n \cos U_n)},$$

$$\delta_m = \frac{MV_m \sin V_m}{(-1)^m q\left[(kH)^2 + V_m^2\right]\sin((1-q)V_m) - M(\sin V_m + V_m \cos U_m)}. \tag{22}$$

Equations (21), (22) give the solution of the spectral problem (5a, c).

5.4. Reconstructing buoyancy frequency in an analytic form for symmetric profiles

By standard techniques, the original problem (5a, c) can be reduced to the solution of an integral equation with a symmetric positive-definite kernel:

$$W(z) = \lambda \int_0^H \left(\mu(\xi) - f^2\right) W(\xi)\, G(\xi, z)\, \mathrm{d}\xi, \ \text{where} \tag{23}$$

$$G(\xi, z) = \begin{cases} \sinh(k\xi)\sinh(k(H-z)), & 0 \le \xi \le z \\ \sinh(kz)\sinh((H-\xi)), & z \le \xi \le H \end{cases}, \quad \lambda = \frac{k}{\sinh(kH)\left(\omega^2 - f^2\right)}. \tag{24}$$

After symmetrization of Equation (23), we apply the theorem on the trace of an integral equation (Mercer's theorem [19, p. 210]). We obtain the equality [3, Equation (3.6.1), p. 239]:

$$\int_0^H \left[\mu(z) - f^2\right] \frac{\sinh(kz)\sinh(k(H-z))}{\sinh(kH)}\, \mathrm{d}z = \frac{1}{k}\sum_j \gamma_j^2(k) = F(k), \ \text{where } \gamma_j^2(k) = \omega_j^2(k) - f^2 \tag{25}$$

Here $\omega_j(k)$ stand for the eigenfrequencies of Problem (5). Boundary-value problem (5) gives the curves $\omega_j^2(k^2)$.

Assume k to be an independent variable. Equality (25) will be interpreted as a first-order Fredholm integral equation. In dimensionless variables this equation has the form:

$$\int_0^1 \varphi(u) L(x, u)\mathrm{d}u = \tilde{F}(x), \ \text{where } x = kH, \quad z = uH, \quad \varphi(u) = \mu(u) - f^2, \tag{26}$$

$$L(x, u) = \frac{\sinh(xu)\sinh(x(1-u))}{\sinh(x)}, \quad \tilde{F}(x) = \frac{1}{x} F\left(\frac{x}{H}\right). \tag{27}$$

Integral equation (19) admits a closed-form solution for the case when the function $\mu(z)$ is symmetrical about the segment centre.

We have $L(x, u) = \dfrac{1}{2\sinh(x)}(\cosh(x) - \cosh(2xu - x))$. Then it follows from (26) that:

$$\frac{\coth(x)}{2}\int_0^1 \varphi(u)\mathrm{d}u - \frac{1}{4\sinh(x)}\int_0^1 \varphi(u)\mathrm{e}^{2x\left(u-\frac{1}{2}\right)}\mathrm{d}u - \frac{1}{4\sinh(x)}\int_0^1 \varphi(u)\mathrm{e}^{-2x\left(u-\frac{1}{2}\right)}\mathrm{d}u = \tilde{F}(x). \tag{28}$$

For the first integral, we replace the integration variable as follows: $u = 1 - v$. Then v will be substituted again for u. Reduction by $\exp(x)$ yields:

$$\int_0^1 (\varphi(u) + \varphi(1-u))\, \mathrm{e}^{-2x\cdot u}\, \mathrm{d}u = \left(C \coth(x) - 2\tilde{F}(x)\right)\left(1 - \mathrm{e}^{-2x}\right), \quad C = \int_0^1 \varphi(u)\, \mathrm{d}u. \tag{29}$$

Let $x \to \infty$ in the last equality. We find: $C = 2\tilde{F}(\infty)$. The right-hand side of the last equality is interpreted here as the Laplace transform of a piecewise continuous function. We obtain [20, Equation (4.2), p. 36]:

$$\varphi(u) + \varphi(1-u) = \frac{1}{i\pi}\int_{C_0 - \mathrm{i}\infty}^{C_0 + \mathrm{i}\infty} \left(\tilde{F}(\infty) - \tilde{F}(x)\right)\left(1 - \mathrm{e}^{-2x}\right)\mathrm{e}^{2x\cdot u}\mathrm{d}x. \tag{30}$$

The function $\mu(z)$ is assumed to by symmetric about the segment centre, *i.e.*, $\varphi(u) = \varphi(1-u)$. Then

$$\varphi(u) = \frac{1}{2i\pi}\int_{C_0 - \mathrm{i}\infty}^{C_0 + \mathrm{i}\infty} \left(\tilde{F}(\infty) - \tilde{F}(x)\right)\left(1 - \mathrm{e}^{-2x}\right)\mathrm{e}^{2x\cdot u}\mathrm{d}x, \quad 0 < u < 1. \tag{31}$$

We take $x = kH, z = uH$ and have:

$$\begin{aligned} &\mu(z) - f^2 = \frac{H}{2i\pi}\int_{C_1 - \mathrm{i}\infty}^{C_1 + \mathrm{i}\infty} \left(\tilde{F}(\infty) - \tilde{F}(kH)\right)\left(1 - \mathrm{e}^{-2k\cdot H}\right)\mathrm{e}^{2k\cdot z}\mathrm{d}k, \quad C_1 = C_0/H, \\ &\tilde{F}(kH) = \frac{1}{kH}\sum_j \left(\omega_j^2 - f^2\right). \end{aligned} \tag{32}$$

Equation (32) determines the function $\boldsymbol{\mu}(z)$ by its eigenfunctions if it is symmetric about the segment centre. For the case of a symmetric buoyancy frequency profile it has been derived in analytical form.

We present the calculation result for the function $\boldsymbol{\mu}(z)$ described in Section 5.3. The eigen-frequencies are obtained by the Equations (21) and (22). The results are presented in Tables 2 and 3.

6. Hydroelastic analogy

6.1. Distribution of elastic-layer density over resonant frequencies obtained from the frequencies of its antiplane variations

Consider the boundary-value problem describing antiplane variations of an elastic layer which is non-homogeneous in thickness. Assume that the layer be bounded by upper and lower

Table 2. The reconstruction accuracy of the function $\boldsymbol{\mu}(z)$ *versus* the number of significant digits (four terms in the trace).

Function $\boldsymbol{\mu}(z)$ reconstruction error in the norm	Number of significant digits in ω and k	
	2	3
In the space C:	16%	12·80%
In the space L_1:	13%	10·60%
In the space L_2:	27%	12·60%

Table 3. The reconstruction accuracy of the function $\boldsymbol{\mu}(z)$ *versus* the number of the terms in the trace for three significant digits in ω and k.

Function $\boldsymbol{\mu}(z)$ reconstruction error in the norm	Number of terms in the trace	
	3	4
In the space C	12·70%	12·50%
In the space L_1	9·90%	8·80%
In the space L_2	11·50%	9·30%

fixed. This studied problem for the amplitude function $W(z)$ is reduced to the following Sturm-Liouville problem:

$$W''(z) + \left(\mu(z)\,\omega^2 - k^2\right) W(z) = 0, \quad W(0) = 0, \quad W(H) = 0, \tag{33}$$

where $\mu(z) = \rho(z)\big/G$; $\rho(z)$ is the law of the thickness distribution of the elastic layer, G is the shear modulus, ω is the eigenfrequency (free-oscillation frequency) and k is the wave number.

Let us represent the function $\mu(z)$ and $W(z)$ as $\mu(z) = \sum_{\mathrm{i}} \mu_i \cos\dfrac{i\pi\cdot z}{H}$, $W(z) = \sum_{\mathrm{i}} W_i \sin\dfrac{i\pi\cdot z}{H}$ and substitute these series in (33). By equating coefficients for similar harmonics, we obtain a system of linear equations in W_i. Let us confine ourselves to a finite number of harmonics and equate the system determinant to zero. We obtain a frequency equation of the form $F\left(\mu_i, k^2, \omega^2\right) = 0$. Assume the coefficients $\boldsymbol{\mu}_i (i = 0, 1, 2)$ to be pre-specified. From the frequency equation we find $\omega_j^2\left(k^2\right) (j = 1, 2, \ldots, J)$.

Pairs of $\omega_j^2\left(k^2\right) (j = 1, 2, \ldots, J)$ are known, even though with a certain error. On substituting these pairs in the frequency equation, we obtain a system of nonlinear equations in $\boldsymbol{\mu}_i$. If the values of the pairs $\omega_j^2\left(k^2\right)$ are taken from different dispersion curves for $j = j_1$ and $j = j_2$, the parameters $\boldsymbol{\mu}_i$ are determined uniquely. If the values of the pairs $\omega_j^2\left(k^2\right)$ are taken from a single dispersion curve, the recovered parameters μ_i will not be unique. Let us study how the accuracy of the set input data affects the accuracy of the obtained function $\mu(z)$. As a specific example let us consider $\mu(z) = A_0 + A_1 \cos\dfrac{\alpha\pi\cdot z}{H}$, with $A_0 = 1$, $A_1 = 0{\cdot}1$, $\alpha = 2{\cdot}01$, $H = 1$.

Table 4 shows the relative error in the metrics L_1 and L_2 used for the reconstruction of the function $\mu(z)$ depending on the number of significant digits specified for ω_j when two harmonics are used.

Table 4. Error in the reconstruction of the function $\mu(z)$ *versus* the number of significant digits set for ω_j.

Number of significant digits	$\varepsilon_2(L_2)\%$	$\boldsymbol{\varepsilon}_1(L_1)\%$
3	0·36	0·2
2	7·42	4·8
1	110	51

Let us analyse the table. The density distribution of the non-homogeneous layer can be reconstructed from two basic resonant frequencies. The accuracy is sufficient with two significant digits in the frequencies.

6.2. Geometrical non-homogeneity of a rod determined from the resonant frequency of its free oscillations

Consider the problem of longitudinal oscillations of a non-homogeneous rod with a variable section [21, Equation (20.1), p. 169]. The rod is fixed at both ends:

$$m\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(EF\frac{\partial u}{\partial x}\right),\quad u(0)=0,\quad u(l)=0. \tag{34}$$

Here $u=u(x,t)$ is the longitudinal displacement of the rod points, $m=m(x)$ is the mass of the rod unit length, $m=\rho F$, $\rho=\rho(x)$ designates the law of material density distribution along the rod, $E=E(x)$ is Young's modulus, $F=F(x)$ is the cross-sectional area. We seek time-periodic solutions: $u\,(x,t)=U(x)\mathrm{e}^{\mathrm{i}\omega t}$. Then, for the amplitude function $U(x)$, we obtain an equation with variable coefficients. Now we introduce the notation, $EF=\psi, m\big/EF=\rho\big/E=1\big/c^2$, $m=m_0+m_1\,(x)\,, m_1=\rho_1 F, m_0=\text{const}, m_0\big/E_0F=\rho_0\big/E_0=1\big/c_0^2$. Here, $c=c\,(x)$ is the local sound velocity for an arbitrary density $\rho(x)$ and for arbitrary Young's modulus $E(x), c_0$ designates the sound velocity for constant density $\rho_0=$const and constant Young's modulus $E=E_0=$const. In the equation for the amplitude function we perform the following substitutions: $U=y\big/\sqrt{\psi}, x=l\xi, y(x)=y(l\xi)=f(\xi)$ and introduce the following notations:

$$\omega^2 l^2\big/c_0^2=\Omega^2,\quad \rho_1\big/\rho_0=\varphi\,(\xi)\,, \tag{35}$$

$$\mu(\xi)=1\big/4\left(\eta'\big/\eta\right)^2-1\big/2\left(\eta''\big/\eta\right),\quad \eta(\xi)=\psi(l\xi)=\psi(x)=EF(x). \tag{36}$$

Then, for the function f at $\rho_1=0$, we obtain the boundary-value problem:

$$f''(\xi)+\left(\mu(\xi)+\Omega^2\right)\,f(\xi)=0,\quad f(0)=0,\quad f(1)=0. \tag{37}$$

Let us show the non-uniqueness of the solution of the inverse problem about obtaining the rod geometrical non-homogeneity from the resonant frequencies of its free oscillations. Indeed, if the function $\mu(\xi)$ is reconstructed by any of the above-mentioned methods from the spectral numbers of Problem (37), then it can be used to find $\eta(\xi)=\psi(x)=EF(x)$ from (36). At the same time, it is possible to add to the obtained function $\eta(\xi)$ also the function $\boldsymbol{\eta}_0\,(\boldsymbol{\xi})$ satisfying the homogeneous equation in (36) at $\mu(\xi)=0:\eta_0(\xi)=(a\xi+b)^2=EF(x)=\left(\frac{a}{l}x+b\right)^2$ where a and b are arbitrary constants.

Note that, if the rod cross-section is taken with such a profile $F(x)$, then the resonant oscillation frequencies of such a rod that is non-homogeneous in length coincide with the resonant frequencies of a homogeneous rod with constant longitudinal cross-section.

6.3. Rod non-homogeneity determined from its bending vibrations

6.3.1. *Problem statement*

In the linear statement, consider the problem about bending vibrations of a pivoted rod. The rod is assumed to be loaded longitudinally by the force $p\,(x)$ [21, Equation (20.1), p. 169]

$$EJ\frac{\partial^2 f}{\partial x^2}=-pf+M. \tag{38}$$

Here f is the road deflection (blend??), $E=E(x)$ is Young's modulus, $J=J(x)$ stands for the moment of inertia of the cross-section with respect to the neutral axis, M is the bending moment of the rod with respect to the principal central axis due to other forces aside from the longitudinal forces $p(x)$; the axis Ox is directed along the rod.

Differentiate Equation (38) twice with respect to x. We obtain [21, Equation (18.1), p. 148]

$$\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 f}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}(pf)-\frac{\partial^2 M}{\partial x^2}=q\,(x,t)\,. \tag{39}$$

where $q\,(x,t)$ is the distributed load on the rod. In the absence of an external active distributed inertial load, $-\rho F\dfrac{\partial^2 f}{\partial t^2}$ is the distributed load [21]. Here, $\rho=\rho(x)$ designates the material density, $F=F(x)$ stands for the cross-sectional area of the rod, and t is the time. Then it follows from (39):

$$\frac{\partial^2}{\partial x^2}\left(EJ\frac{\partial^2 f}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}(pf)=-\rho F\frac{\partial^2 f}{\partial t^2}. \tag{40}$$

Boundary conditions for a pivoted rod have the form [21, p. 150]

$$f(0)=0,\qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0}=0,\qquad f(l)=0,\qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=l}=0. \tag{41}$$

The origin of coordinates is taken to be at the left-hand end of the beam.

Impose the condition of time periodicity

$$f\left(t+\frac{2\pi}{\omega}\right)=f\,(t)\,. \tag{42}$$

Here $\omega=\dfrac{2\pi}{T}$ is the frequency oscillation, T is the oscillation period.

The solution of Equation (40) is sought as

$$f\,(x,t)=y\,(x)\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{43}$$

For the function $y(x)$, we obtain the following boundary-value problem [21, p. 148]:

$$\begin{gathered}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(E(x)J(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}(p\,(x)\,y)=\rho(x)F(x)\omega^2 y,\\ y\,(0)=0,\quad y''\,(0)=0,\quad y\,(l)=0,\quad y''\,(l)=0.\end{gathered} \tag{44}$$

The $\{\omega_j\}$, $j=1,2,\ldots,$ will be resonant frequencies of bending (flexural) vibrations of a non-homogenous rod. The rod non-homogeneity is due either to the geometrical non-homogeneity ($J(x)$, $F(x)$), or the physical one ($E(x)$, $\rho(x)$), or to the non-homogeneity of the longitudinal force $p(x)$.

The following problem is now stated: by making use of the known resonant frequencies ω_j, how can we determine the character of the rod non-homogeneity?

6.3.2. *Problem statement for the particular case*

Let $E(x) = E = \text{const}$, $J(x) = J = \text{const}$, $p(x) = 0$, and $\rho(x)F(x) = m(x)$ be the linear mass of the rod. Represent $m(x)$ as $m(x) = m_0 + m_1(x), m_0 = \text{const}$. Here $m_0 = \rho_0 F$ is the linear mass of a homogeneous rod with density ρ_0, and $m_1(x)$ is the sought linear mass of non-homogeneity along the rod.

Then we have

$$\frac{d^4y}{dx^4} - \alpha^4 y = \frac{m_1(x)}{m_0}\alpha^4 y \equiv \Phi(x), \qquad \alpha^4 = \frac{m_0\omega^2}{EJ} = \text{const},$$
$$y(0) = 0, \quad y''(0) = 0, \quad y(l) = 0, \quad y''(l) = 0. \tag{45}$$

6.3.3. *Derivation of the frequency equation*

Denote the right-hand side of Equation (45) by the function $\Phi(x)$. Let us temporarily consider it to be known. Equation (45) is solved by the method of variation of arbitrary constants and yields:

$$y(x) = \int_0^1 \frac{m_1(\xi)}{m_0} G(x, \xi, \alpha) y(\xi) d\xi, \tag{46}$$

$$G(x, \xi, \alpha) = \frac{\alpha}{2}\left[\frac{G_i(x,\xi)}{\sin\alpha} - \frac{G_h(x,\xi)}{\sinh\alpha}\right]. \tag{47}$$

Here,

$$G_i(x,\xi) = \begin{cases} \sin(\alpha\xi)\sin(\alpha(l-x)), & 0 \le \xi \le x \\ \sin(\alpha x)\sin(\alpha(l-\xi)), & x \le \xi \le l \end{cases}, \tag{48}$$

and

$$G_h(x,\xi) = \begin{cases} \sinh(\alpha\xi)\sinh(\alpha(l-x), & 0 \le \xi \le x \\ \sinh(\alpha x)\sinh(\alpha(l-\xi)), & x \le \xi \le l \end{cases}. \tag{49}$$

We write out the integrals in Equation (46) using arbitrary quadrature formulae and assume that the discrete values of x coincide with Gaussian nodes for (46), $x_n = \xi_n$. We obtain a system of homogeneous equations in $y(\xi_n)$. The determinant of this system is a frequency equation of the bending vibrations of the rod with arbitrarily distributed mass along it.

Consider the case of a localized mass m_1:

$$\frac{m_1(x)}{m_0} = A\delta(x - x_1). \tag{50}$$

where A is a dimensionless constant, $\delta(x)$ is the Dirac function, and x_1 is the coordinate of the localized mass.

On substituting (50) in (46) we obtain

$$y(x) = AG(x, x_1, \alpha)y(x_1). \tag{51}$$

Assume $x = x_1$. Reduce both sides by $y(x_1)$. We derive the frequency equation for this particular case as follows:

$$1 = A\frac{\alpha}{2}\left[\frac{\sin(\alpha x_1)\sin(\alpha(1-x_1))}{\sin\alpha} - \frac{\sinh(\alpha x_1)\sinh(\alpha(1-x_1))}{\sinh\alpha}\right], \tag{52}$$

or

$$2\sinh\alpha \ \sin\alpha = A\alpha\left[\sin(\alpha x_1)\sin(\alpha(1-x_1))\sinh\alpha - \sinh(\alpha x_1)\sinh(\alpha(1-x_1))\sin\alpha\right]. \quad (53)$$

Equation (53) is the frequency equation of the bending vibrations of the rod with the localized mass at the point x_1 with no allowance made for the longitudinal force.

6.3.4. *Solution of the frequency equation*

From Equation (53) we derive asymptotic formulae for the roots. Let $A \to 0$. Then, for $x_1 \to 0$, it follows from (53) that

$$\omega_n \approx \frac{1}{l^2}\sqrt{\frac{EJ}{m_0}}\pi^2 n^2\left(1 - A\pi^2 n^2 x_1^2\right), \qquad A\pi^2 n^2 x_1^2 \ll 1. \quad (54)$$

For the case $x_1 \to 1$ in Equation (54) x_1 should be substituted for $(1-x_1)$. Let $A \to \infty$. Then we have from (53) that

$$\omega_1 \approx \frac{1}{l^2 x_1(1-x_1)}\sqrt{\frac{3EJ}{Am_0}}, \quad x_1 \ge \delta > 0, \quad 1-x_1 \ge \delta > 0, \quad \delta = \text{const}. \quad (55)$$

6.3.5. *Solution of the inverse problem*

In the general case, the localized-mass position is unknown. Then, from two base frequencies ω_1 and ω_2, two values α_1 and α_2 are obtained. They are successively substituted in Equation (53). Then the obtained equalities are divided, which yields an equation for locating the position x_1 of the localized mass.

$$\frac{\sinh\alpha_1 \sin\alpha_1}{\sinh\alpha_2 \sin\alpha_2} = \frac{\alpha_1}{\alpha_2}\frac{\sin(\alpha_1 x_1)\sin(\alpha_1(1-x_1))\sinh\alpha_1 - \sinh(\alpha_1 x_1)\sinh(\alpha_1(1-x_1))\sin\alpha_1}{\sin(\alpha_2 x_1)\sin(\alpha_2(1-x_1))\sinh\alpha_2 - \sinh(\alpha_2 x_1)\sinh(\alpha_2(1-x_1))\sin\alpha_2}. \quad (56)$$

From Equation (56), x_1 is found. In doing this, we only select values from $0 \le x_1 \le 1$. From the obtained value x_1 from (53) A is determined for $\alpha = \alpha_1$ or $\alpha = \alpha_2$. Since Equation (56) is transcendental in x_1, the solution of the inverse problem may prove to be non-unique. In order to select a unique solution, either supplementary conditions should be imposed, or different frequencies should be set and the following functional should be minimized with respect to the parameters A and x_1:

$$\Psi(x_1, A) = \sum_{n=1}^{N}\left(\Psi_{1n} - A\Psi_{2n}(x_1)\right)^2, \text{ where, } \Psi_{1n} = \sin\alpha_n, \quad (57)$$

and

$$\Psi_{2n} = \frac{\alpha_n}{2}\left[\sin(\alpha_n x_1)\sin(\alpha_n(1-x_1)) - \frac{\sin\alpha_n \sinh(\alpha_n x_1)\sinh(\alpha_n(1-x_1))}{\sinh\alpha_n}\right]. \quad (58)$$

Here, N is the number of known resonant frequencies, $\alpha_n = \omega_n l^2\sqrt[4]{\frac{m_0}{EJ}}$.

From the equation $\frac{\partial\Psi(x_1, A)}{\partial A} = 0$ we obtain an expression for the parameter A as a function of x_1:

$$A \equiv A(x_1) = \sum_{n=1}^{N}\Psi_{1n}\Psi_{2n}(x_1) \Bigg/ \sum_{n=1}^{N}\Psi_{2n}^2(x_1). \quad (59)$$

By substituting this value of A in the equation $\frac{\partial \Psi(x_1, A)}{\partial x_1} = 0$, we finally obtain the following equation from which to obtain the points x_1:

$$A(x_1) \sum_{n=1}^{N} \Psi'_{2n}(x_1) \left[\Psi_{1n}(x_1) - A(x_1) \Psi_{2n}(x_1) \right] = 0. \tag{60}$$

By means of (60), from the obtained x_1, the corresponding values of A are calculated.

In Tables 5 and 6 the accuracy at which the parameters A are obtained and x_1 are given *versus* the accuracy of the input data. The accurate values are $x_1 = 0{\cdot}1$, $A = 1$. From Tables 5 and 6, making use of two frequencies with two significant digits, we may calculate the parameter A to an accuracy of up to 12·87%. The parameter x_1 is found to an accuracy of up to 11·52%. Let three frequencies be specified to the same accuracy. The above-mentioned parameters are determined to an accuracy of up to 2·04% and 0·31%, respectively.

7. Conclusion

This work presents new methods for the solution of inverse Sturm-Liouville problems. The suggested methods make it possible to determine structural non-homogeneity in the ocean column and defects in certain elements of engineering structures.

The oceanological problems are solved using remote sensing from Artificial Earth Satellites for investigating the free ocean surface and studying glitters on it. The glitter rate is used to determine the spectrum of the internal waves. A detailed description of the procedure is given in [2].

In our work, ocean non-homogeneity is determined by a study of the internal wave spectrum. Such non-homogeneities may be bathyscaphes, fish shoals, submarines, scuba divers, etc. The detection of fish shoals can significantly reduce the costs incurred by fishing vessels looking for fish shoals.

Detecting imperfections and defects of engineering constructions will first provide adequate quality control and, secondly, can help to check the structure of elements out of view. The same methods can be employed to check baggage during customs inspections and security checks at airports. The same methods can replace the X-ray for examination of osseous

Table 5. Reconstruction of the parameters x_1 and A from two frequencies.

Number of significant digits	x_1 (error %)	A (error %)
1	0·179216 (79·22%)	0·620922 (37·91%)
2	0·111519 (11·52%)	0·871245 (12·87%)
3	0·100547 (0·55%)	0·992408 (0·76%)

Table 6. Reconstruction of the parameters x_1 and A from three frequencies.

Number of significant digits	x_1 (error %)	A (error %)
1	0·105603 (5·60%)	1·075803 (7·58%)
2	0·099693 (0·31%)	1·020441 (2·04%)
3	0·99972 (0·03%)	1·000714 (0·07%)

tissues to detect fissure fractures and sarcomas. These methods make it possible to monitor a patient's condition without exposure to radiation.

The topic calls for further investigation and generalization. We deem it necessary to study inverse problems for membranes, shells, composite and viscoelastic media. Inverse problems should be solved for media contacting other media, *e.g.* liquid, composite and viscoelastic. It is also necessary to consider horizontal non-homogeneity. The work can also be further developed mathematically. It is necessary to transfer from particular problems treated here to a general formulation of these problems. Topics such as the uniqueness of the problem solution are worth our attention, which also requires formulating (preferably in a general form) supplementary conditions required for it. The conditions should also be investigated under which the inverse spectral problems cannot be solved.

Of interest is also the study of dynamic inverse problems consisting in the determination of the medium structure from its response to external disturbances. Nonlinear inverse problems are also worth considering.

Upon some further development the suggested methods could be applied to studying blood circulation in vessels to detect thrombus formation. This method could be applied to the examination of vascular walls checking for hemoliths.

These methods can be useful for controlling medium non-homogeneity, motion of bodies in fluid and gas, laws of deformation of elastic bodies and bodies with more complicated rheological properties.

The solution of control problems will make it possible to send a probing weather balloon aloft to a pre-specified altitude. Solving the problems of non-destructive control will make it possible to provide body-strain limitations. The pre-specified law of flow past bodies will help to reduce the body resistance and drag during motion in the fluid or air.

References

1. B.M. Levitan, *Inverse Sturm-Liouville Problems*. Moscow: Nauka (1967) 240 pp. (in Russian).
2. B.A. Nelepo, G.K. Korotaev, V.S. Suetin and Yu.V. Terehin, *Aerospace Radar Imaging of Ocean*. Kiev: Naukova Dumka (1985) 168 pp. (in Russian).
3. E.N. Potetunko, L.V. Cherkesov, D.S. Shubin and Ye.N. Scherback, *Free Vibrations and Inverse Spectral Problems*. Moscow: Vuzovskaya Kniga (2001) 288 pp. (in Russian).
4. Ye.N. Scherback, Confidence interval of spectrum of fundamental frequencies of stratified fluid. In: I.I. Vorovich: Proceedings of IV International Conference *Contemporary Problems of Continuum Mechanics*. Rostov-on-Don, NCSCHS (1998) pp. 218–221 (in Russian).
5. V.V. Ryndina, Some questions of reconstruction uniqueness of the Brunt-Vaisala frequency by disperse curves. *J. Appl. Math. Mech.* 64(1), (2000) 121–126 (in Russian).
6. L.V. Cherkesov, E.N. Potetunko and D.S. Shubin, Reconstruction of ocean density distribution from its wave spectrum. *Int. J. Fluid Mech. Res.* 30 (2003) 11–23.
7. Ye.A. Anosova and V.V. Trepachev, Estimation of water density variation induced by man-caused pollution. In: V.L. Gaponov: Collected Scientific Proceedings *Safety of Vital Functions. Protection of Labour and Environment*. RSASHM, Rostov-on-Don, Issue 7 (international) (2003) pp. 82–84 (in Russian).
8. W. Rundell and P.E. Sacks, The reconstruction of Sturm-Liouville operators. *Inverse Problems* 8 (1992) 457–482.
9. B.M. Brown, V.S. Samko, I.W. Knowles and M. Marletta, Inverse spectral problem for the Sturm-Liouville equation. *Inverse Problems* 19 (2003) 235–252.
10. Patrick J. Browne and B.D. Sleeman, A uniqueness theorem for inverse eigenparameter dependent Sturm-Liouville problems. *Inverse Problems* 13 (1997) 1453–1462.
11. Chao-Liang Shen, Some inverse spectral problems for vectorial Sturm-Liouville equations. *Inverse Problems* 17 (2001) 1253–1294.

12. R.O. Hryniv and Ya.V. Mykytyuk, Inverse spectral problems for Sturm-Liouville operators with singular potentials. *Inverse Problems* 19 (2003) 665–684.
13. V. Pivovarchik and C. van der Mee, The inverse generalized Regge problem. *Inverse Problems* 17 (2001) 1831–1845.
14. V.B. Daskalov, On the inverse problem for the regular Sturm-Liouville operator. *Inverse Problems* 12 (1996) 207–217.
15. Yu.Z. Miropolsky, *Dynamics of Internal Gravitational Waves in Ocean*. Leningrad: Hydrometeoizdat (1981) 302 pp. (in Russian).
16. H. Bateman and A. Erdelyi, *Higher Transcendental Functions*. Vol. 2. Moscow: Nauka (1966) 296 pp. (in Russian).
17. A. Erdelyi, *Asymptotic Decompositions*. Moscow: *Phys. Math. Lit.* (1962) 128 pp. (in Russian).
18. N.I. Moiseev, *Asymptotic Methods of Non-Linear Mechanics*. Moscow: Nauka (1973) 351 pp. (in Russian).
19. M.L. Krasnov, *Integral Equations*, Moscow: Nauka (1975) 304 pp. (in Russian).
20. G. Doetsch, *Guide to the Applications of Laplace Transforms*, Moscow: Nauka (1965) 286 pp. (in Russian).
21. V.L. Biderman, *Applied Theory of Mechanical Oscillations. Tutorial for Institutes of Technology*. Vysshaya Shkola. (1972) 416 pp. (in Russian).